About sum rules for Gould-Hopper polynomials

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Abstract

We show that various identities from [1] and [3] involving Gould-Hopper polynomials can be deduced from the real but also complex orthogonal invariance of multivariate Gaussian distributions. We also deduce from this principle a useful stochastic representation for the inner product of two non-centered Gaussian vectors and two non-centered Gaussian matrices.

Résumé

Formules de sommation pour les polynômes de Gould et Hopper Nous montrons que des identités pour les polynômes de Gould et Hopper issues de [1] et [3] peuvent être déduites de l'invariance orthogonale réelle mais aussi complexe des lois Gaussiennes multivariées. Nous déduisons aussi de cette propriété une représentation stochastique du produit scalaire de deux vecteurs gaussiens non-centrés, ainsi que de deux matrices gaussiennes non-centrées.

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1 An identity by Graczyck and Nowak

We adopt the multi-index notation: with $n \geq 1$, a multi-index is denoted as $\underline{\mathrm{m}} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and its length $|\underline{\mathrm{m}}| = m_1 + \ldots + m_n$. In particular, the multi-index factorial is $\underline{\mathrm{m}}! = m_1! \ldots m_n!$ and, with $\underline{\mathrm{x}} \in \mathbb{R}^n$, the multi-index power $\underline{\mathrm{x}}^{\underline{\mathrm{m}}} = x_1^{m_1} \ldots x_n^{m_n}$. The Gould-Hopper polynomials [2] are

$$g_m(x, p) = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{m!}{k!\,(m-2k)} p^k x^{m-2k}; \;\; x, p \in \mathbb{R}. \tag{1}$$

Their multi-index version reads $g_{\underline{\mathrm{m}}}(\underline{\mathrm{x}}, p) = \prod_{i=1}^n g_{m_i}(x_i, p)$.

We use the following notations: $\underline{\mathrm{X}} \sim \mathcal{N}(\underline{\mathrm{m}}_X, \underline{\underline{R_X}})$ expresses the fact that the vector $\underline{\mathrm{X}}$ is Gaussian with mean $\underline{\mathrm{m}}_X$ and covariance matrix $\underline{\underline{R_X}}$; the expectation operator is denoted as $\mathbb{E}$, the Euclidean norm of vector $\underline{\mathrm{z}}$ as $|\underline{\mathrm{z}}|$ and $(n)_j$ is the Pochhammer symbol $\frac{\Gamma(n+j)}{\Gamma(n)}$. Finally, underlined variables denote vectors, random variables are capitalized, $\underline{\mathrm{x}}^t$ denotes the transpose vector of $\underline{\mathrm{x}}$ and $\sim$ means equality in distribution.

Our first result is a stochastic representation of the inner product of two non centered Gaussian vectors.

Theorem 1.1 *If $\underline{x}$, $\underline{y} \in \mathbb{R}^n$, $p \in \mathbb{R}$ and $\underline{X}$, $\underline{Y}$ are independent random vectors $\sim \mathcal{N}(\underline{0}, \underline{\underline{I_n}})$ then*

$$\left(\underline{x} + \sqrt{p}\underline{N}\right)^t \left(\underline{y} + \sqrt{p}\underline{M}\right) \sim \left(x + \sqrt{p}N_1\right)\left(y + \sqrt{p}M_1\right) + pZ_{n-1}N \tag{2}$$

where N, N_1 and M_1 are independent standard Gaussian random variables,

$$x = \frac{|\underline{x} + \underline{y}| + |\underline{x} - \underline{y}|}{2}, \quad y = \frac{|\underline{x} + \underline{y}| - |\underline{x} - \underline{y}|}{2} \tag{3}$$

and Z_{n-1} is chi-distributed with $n-1$ degrees of freedom and independent of

N, N_1 and M_1.

Proof 1.2 *The proof is based on the polarization identity, that allows to express this inner product as*

$$\frac{1}{4}\left[|\underline{x}+\sqrt{p}\underline{N}+\underline{y}+\sqrt{p}\underline{M}|^2-|\underline{x}+\sqrt{p}\underline{N}-\underline{y}-\sqrt{p}\underline{M}|^2\right]$$
$$\sim\frac{1}{4}\left[|\underline{x}+\underline{y}+\sqrt{2p}\underline{\tilde{N}}|^2-|\underline{x}-\underline{y}+\sqrt{2p}\underline{\tilde{M}}|^2\right]$$

where vectors $\underline{\tilde{M}}=\frac{1}{\sqrt{2}}\left(\underline{N}+\underline{M}\right)$ and $\underline{\tilde{N}}=\frac{1}{\sqrt{2}}\left(\underline{N}-\underline{M}\right)$ are again Gaussian and independent as a consequence of the orthogonal invariance.

The next step is again a consequence of this invariance: if $\underline{G}\sim\mathcal{N}(\underline{0},\underline{\underline{I_n}})$ and $\underline{z}\in\mathbb{R}^n$, then the norm of the random vector $\underline{z}+\underline{G}$ depends on $|\underline{z}|$ only; more precisely

$$|\underline{z}+\underline{G}|\sim||\underline{z}|\underline{e}_1+\underline{G}|=\sqrt{\left(|\underline{z}|+G_1\right)^2+G_2^2+\ldots+G_n^2}$$

where $\underline{\boldsymbol{e}}_1$ is the first (or any) column vector of the identity matrix $\underline{\underline{I_n}}$. We deduce that

$$\left(\underline{x}+\sqrt{p}\underline{N}\right)^t\left(\underline{y}+\sqrt{p}\underline{M}\right)\sim\frac{1}{4}\left[||\underline{x}+\underline{y}|\underline{e}_1+\sqrt{p}\underline{N}+\sqrt{p}\underline{M}|^2-||\underline{x}-\underline{y}|\underline{e}_1+\sqrt{p}\underline{N}-\sqrt{p}\underline{M}|^2\right].$$

and by the polarization identity, with X and Y as in Theorem 1.1, this expression simplifies to

$$\left(x\underline{e}_1+\sqrt{p}\underline{N}\right)^t\left(y\underline{e}_1+\sqrt{p}\underline{M}\right)=\left(x+\sqrt{p}N_1\right)\left(y+\sqrt{p}M_1\right)+p\sum_{i=2}^{n}N_iM_i.$$

The proof follows from the identity in distribution, with $Z_{n-1}\sim\chi_{n-1}$ independent of $N\sim\mathcal{N}\left(0,1\right)$:

$$\sum_{i=2}^{n}N_iM_i\sim\left(\sum_{i=2}^{n}M_i^2\right)^{\frac{1}{2}}N=Z_{n-1}N$$

We deduce an elementary proof of the following [3, Thm 3], originally derived using generating functions.

Theorem 1.3 *For all $M \in \mathbb{N}$, $p \in \mathbb{R}$ and $\underline{x}, \underline{y} \in \mathbb{R}^n$, we have, with x and y as in (3),*

$$\sum_{|\underline{m}|=M} \frac{1}{\underline{m}!} g_{\underline{m}}(\underline{x}, p)\, g_{\underline{m}}(\underline{y}, p) = \sum_{j=0}^{\lfloor M/2 \rfloor} \frac{(2p)^{2j}}{j!\,(M-2j)!} \left(\frac{n-1}{2}\right)_j g_{M-2j}(x,p)\, g_{M-2j}(y,p) \tag{4}$$

Proof 1.4 *We use the moment representation of Gould-Hopper polynomials*

$$g_m(x,p) = \mathbb{E}\left(x + \sqrt{2p}N\right)^m \tag{5}$$

where $N \sim \mathcal{N}(0,1)$: it is a consequence of definition (1) and of the fact that $\mathbb{E}N^{2k} = \frac{(2k)!}{2^k k!}$ and $\mathbb{E}N^{2k+1} = 0$. We then recognize the left-hand side of (4) as the multinomial expansion

$$\begin{aligned} &\mathbb{E} \sum_{|\underline{m}|=M} \frac{1}{\underline{m}!} \prod_{i=1}^n \left(x_i + \sqrt{2p}N_i\right)^{m_i} \left(y_i + \sqrt{2p}M_i\right)^{m_i} \\ &= \frac{1}{M!} \mathbb{E}\left(\sum_{i=1}^n \left(x_i + \sqrt{2p}N_i\right)\left(y_i + \sqrt{2p}M_i\right)\right)^M \end{aligned}$$

where $\underline{M}$ and $\underline{N} \sim \mathcal{N}\left(0, \underline{\underline{I_n}}\right)$ are independent. The sum is identified as the inner product

$$(\underline{x} + \sqrt{2p}\underline{N})^t (\underline{y} + \sqrt{2p}\underline{M}).$$

Using the multinomial expansion of the $M-th$ power of the stochastic equivalent (2) of this expression and taking expectations with $\mathbb{E}Z_{n-1}^{2j} = 2^j \left(\frac{n-1}{2}\right)_j$, we obtain the desired result.

2 Two identities by Daboul and Mizrahi

The group of complex rotations $\mathcal{O}_{\mathbb{C}}(n)$ is the set of $n \times n$ complex matrices $\underline{\underline{O}}$ such that $\underline{\underline{O}}\,\underline{\underline{O}}^t = \underline{\underline{O}}^t\,\underline{\underline{O}} = \underline{\underline{I_n}}$. The **real** inner product $\underline{x}^t\,\underline{y} = \sum_{i=1}^n x_i y_i,\ \underline{x},\ \underline{y} \in \mathbb{C}^n$, is preserved under the action of this group. In [1], sum rules for Gould-Hopper polynomials are proved using the covariant transformation property under $\mathcal{O}_{\mathbb{C}}(n)$ of the raising operators associated with these polynomials. We show here that these sum rules can be equivalently deduced from the following complex rotational invariance principle.

Theorem 2.1 *For any $\underline{\underline{O}} \in \mathcal{O}_{\mathbb{C}}(n)$ and $\underline{N} \sim \mathcal{N}(0, \underline{\underline{I_n}})$, the Gould-Hopper polynomial of degree m reads*

$$g_m(x,p) = \mathbb{E}\left(x + \sqrt{2p}\left(\underline{\underline{O}}\,\underline{N}\right)_j\right)^m, \ \forall j \in [1,n].$$

Proof 2.2 *We only need to check that $\left(\underline{\underline{O}}\,\underline{N}\right)_j \sim \mathcal{N}(0,1)$; but, using the fact that the identity $\mathbb{E}\exp(2\imath t N_j) = \exp(-t^2)$ holds for any **complex** value of t, the characteristic function of this random variable reads*

$$\mathbb{E}\exp\left(\imath t\left(\underline{\underline{O}}\,\underline{N}\right)_j\right) = \exp\left(-t^2\sum_{k=1}^n O_{jk}^2\right) = \exp\left(-t^2\right)$$

by the complex orthogonality condition.

Using this probabilistic invariance, we derive a simple proof of the summation theorem [1, Prop. 1]

Theorem 2.3 *For any $\underline{\underline{O}} \in \mathcal{O}_{\mathbb{C}}(n)$,*

$$g_m\left(\left(\underline{\underline{O}}\,\underline{x}\right)_i, p\right) = \sum_{|\underline{m}|=m}\binom{m}{m_1,\dots,m_n}\prod_{j=1}^n O_{ij}^{m_j}\, g_{m_j}(x_j,p)$$

Proof 2.4 *The proof is based on the complex rotational invariance (Theorem*

2.1) as follows

$$g_m\left(\left(\underline{\underline{Q}}\,\underline{x}\right)_i, p\right) = \mathbb{E}\left(\left(\underline{\underline{Q}}\,\underline{x}\right)_i + \sqrt{2p}\left(\underline{\underline{Q}}\,\underline{N}\right)_i\right)^k \sim \mathbb{E}\left(\sum_{j=1}^{n} O_{ij}\left(x_j + \sqrt{2p}N_j\right)\right)^m.$$

The desired result is obtained by expanding this power using the multinomial formula as

$$\mathbb{E}\sum_{|\underline{m}|=m}\binom{m}{m_1,\ldots,m_n}\prod_{j=1}^{n} O_{ij}^{m_j}\left(x_j+\sqrt{2p}N_j\right)^{m_j} = \sum_{|\underline{m}|=m}\binom{m}{m_1,\ldots,m_n}\prod_{j=1}^{n} O_{ij}^{m_j} g_{m_j}\left(x_j, p\right).$$

We now prove equally easily the following generalized factorization sum rule [1, Prop.3]

Theorem 2.5 *With $p \in \mathbb{R},\ c, s, x, y \in \mathbb{C}$ and $c^2 + s^2 = 1$,*

$$g_{m_1}\left(cx - sy, p\right) g_{m_1}\left(sx + cy, p\right) = \sum_{r=0}^{m_1+m_2} C_{m_1,m_2,r}\left(c, s\right) g_r\left(x, p\right) g_{m_1+m_2-r}\left(y, p\right)$$

where coefficients $C_{m_1,m_2,r}\left(c,s\right)$ are given by

$$C_{m_1,m_2,r}\left(c,s\right) = \sum_{l=0}^{m_2\wedge r}\binom{m_1}{r-l}\binom{m_2}{l}\left(-1\right)^{m_1-r+l} c^{m_2+r-2l} s^{m_1-r+2l}. \tag{6}$$

Proof 2.6 *By the complex orthogonal invariance, $[N_1, N_2] \sim \mathcal{N}\left(0, I_2\right) \sim [cN_1 - sN_2, sN_1 + cN_2]$ so that*

$$\begin{aligned}
&g_{m_1}\left(cx - sy, p\right) g_{m_2}\left(sx + cy, p\right)\\
&\quad= \mathbb{E}\left(cx - sy + \sqrt{2p}\left(cN_1 - sN_2\right)\right)^{m_1} \mathbb{E}\left(sx + cy + \sqrt{2p}\left(sN_1 + cN_2\right)\right)^{m_2}\\
&\quad= \mathbb{E}\sum_{k=0}^{m_1}\sum_{l=0}^{m_2}\binom{m_1}{k}\binom{m_2}{l}\left(-1\right)^{m_1-k} c^{m_2+k-l} s^{m_1+l-k}\left(x+\sqrt{2p}N_1\right)^{l+k}\left(y+\sqrt{2p}N_2\right)^{m-l+n-k}\\
&\quad= \sum_{r=0}^{m_1+m_1} C_{m_1,m_2,r}\left(c,s\right) g_r\left(x,p\right) g_{m_1+m_2-r}\left(y,p\right)
\end{aligned}$$

with coefficients $C_{m_1,m_2,r}\left(c,s\right)$ as in (6), which coincides with [1, eq. 19].

We note that all former sum rules apply to the Hermite polynomials since $H_n(x) = g_n\left(2x, \sqrt{-2}\right)$. Moreover, theorem 1.1 extends to the matrix case as follows, with $\|\underline{\underline{X}}\|_F$ denoting the Frobenius norm:

Theorem 2.7 *If $\underline{\underline{x}}, \underline{\underline{y}} \in \mathbb{R}^{m\times n}$ and $\underline{\underline{M}}, \underline{\underline{N}} \in \mathbb{R}^{m\times n}$ are independent matrices with i.i.d. standard complex Gaussian entries then, with $M_1, N_1, N \sim \mathcal{N}(0,1)$ and $Z_{mn-1} \sim \chi_{mn-1}$ independent,*

$$\mathtt{tr}(\underline{\underline{x}} + \underline{\underline{N}})^t(\underline{\underline{y}} + \underline{\underline{M}}) \sim (x + N_1)(y + M_1) + Z_{mn-1}N$$

with $x = 1/2(\|\underline{\underline{x}} + \underline{\underline{y}}\|_F^2 + \|\underline{\underline{x}} - \underline{\underline{y}}\|_F^2)$, $y = 1/2(\|\underline{\underline{x}} + \underline{\underline{y}}\|_F^2 - \|\underline{\underline{x}} - \underline{\underline{y}}\|_F^2)$.

Proof 2.8 *Denoting $\underline{x_i}$ the $i-$th column vector of $\underline{\underline{x}}$ and $\underline{u}, \underline{v}$ the vectors with components $u_i, v_i = 1/2(\|\underline{x_i}+\underline{y_i}\|^2 \pm \|\underline{x_i}-\underline{y_i}\|)$, we deduce that $1/2\left(\|\underline{u}+\underline{v}\|^2 \pm \|\underline{u}-\underline{v}\|\right) = x, y$ and by theorem 1.1*

$$\mathtt{tr}(\underline{\underline{x}} + \underline{\underline{N}})^t(\underline{\underline{y}} + \underline{\underline{M}}) = \sum_{i=1}^{m}(\underline{x_i} + \underline{N_i})^t(\underline{y_i} + \underline{M_i}) \sim \sum_{i=1}^{m}(u_i + N_i)(v_i + M_i) + \chi_{n-1}^{(i)}G_i.$$

Using again theorem 1.1, the first sum is distributed as $(x + N_1)(y + M_1) + \chi_{m-1}G$ and since $\chi_{m-1}G_2 + \sum_{i=1}^{m}\chi_{n-1}^{(i)}G_i \sim \chi_{nm-1}N$, the result follows.

References

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